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INDUCTION ON CLOSED, BOUNDED-BELOW SUBSETS OF [illegible]

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ABSTRACT. [illegible] K is closed and bounded below. The induction theorem is presented and [illegible] differential inequalities. [illegible]

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INTRODUCTION

Applications of mathematical induction permeate many classical mathemat- [illegible] [illegible] here for this statement is $\forall n \in I\!N\ (Q(n))$, where "$Q(n)$" means "$Q(n)$ is true". Two common forms of induction are "weak" and "strong" (or "complete") induction (Mendelson [2]):

Weak Induction: If $Q(1)$ and for all $k \in I\!N$, $Q(k+1)$ [illegible]

$$\forall n \in I\!N\ \ (Q(n)).$$

Weak Induction implies Strong Induction:

Strong Induction: If for all $n \in I\!N$, $Q(n)$ holds whenever $Q(m)$ for all $m \in I\!N$ such that $m < n$, then $\forall n \in I\!N (Q(n))$.

Induction works for the natural numbers because of special ordering properties on $\mathbb{N}$ which ensure that each $n \in \mathbb{N}$ has a next largest (successor) element $n + 1$. The Property of Transfinite Induction (see e.g. [illegible]) [illegible] for which every nonempty subset has a least element).

Property of Transfinite Induction: Let $<$ be a well-ordering for the set W. Suppose that for all $w \in W$, we have $Q(w)$ whenever $Q(y)$ for all $y \in W$ such that $y < w$. Then $\forall w \in W(Q(w))$.

Proof: Define $\tilde{Q}(w)$ as the statement "not $Q(w)$". Suppose that for some $z \in W$, $\tilde{Q}(z)$. Then the truth set $\{z \in W | \tilde{Q}(z)\}$ is a nonempty subset of W, and hence contains a least element w. Thus, for all $y < w$, $Q(y)$, and so $Q(w)$, which is a contradiction. □

In general, induction cannot be used to establish results of the form $\forall x \in \mathbb{R}(Q(x))$ because the set of real numbers $\mathbb{R}$ is not well-ordered under the usual order [illegible] holds whenever $y < 3$ for all $y < x$, it is not the case that $x < 3$ for all $x \in \mathbb{R}$.

There are, however, ways to use induction on certain closed subsets of $\mathbb{R}$ for certain predicates Q.

Theorem 1: *Let $a \in \mathbb{R}$. Suppose the truth set $A = \{t \in \mathbb{R} | Q(t)\}$ is an open set in $\mathbb{R}$ and, for all $t \in [a, \infty)$, $Q(t)$ whenever $Q(x)$ for all $x \in [a, t)$. Then*

$$\forall t \in [a, \infty)(Q(t)).$$

Proof: Suppose that $\tilde{Q}(t)$ for some $t \in [a, \infty)$. Then the set $(\mathbb{R} - A) \cap [a, \infty)$ is nonempty, closed, and bounded below, and thus contains its infimum [illegible] and so [illegible], which is a contradiction.

□

Note that the second hypothesis of Theorem 1 vacuously implies $Q(a)$ [illegible]

Theorem 1 can [illegible] and bounded below.

Theorem 2: *Suppose $K \subset \mathbb{R}$ is closed and bounded below in $\mathbb{R}$, $A = \{t \in \mathbb{R} | Q(t)\}$ is open in $\mathbb{R}$, and for all $t \in K$, $Q(t)$, whenever $Q(x)$ for all $x \in K$ such that $x < t$. Then*

$$\forall t \in K(Q(t)).$$

2. APPLICATIONS

To illustrate applicable directions in which induction on closed, bounded below subsets of $I\!R$ may be utilized, we focus on some classical problems [illegible] integral inequalities. Theorem 1 can be used to give [illegible] for integral and delay differential inequalities. These comparison theorems are commonly demonstrated by contradiction, with each proof reiterating a particular incarnation of the proof of Theorem 1. Besides abstracting a method of direct proof [illegible]

[illegible]

[illegible] integral inequalities):

[illegible] Let $J = [a, \infty)$, $K \in C[J \times J \times I\!R, I\!R]$, $x, y, f \in C[J, I\!R]$, and $K(s, t, x)$ be strictly increasing in x for each [illegible]

[illegible]

iii) $x(a) < y(a)$,

[illegible]

$$x(t) < y(t) \quad \forall t \geq a.$$

Proof: Continuously extend x and y to $I\!R$ by defining $x(t) = x(a)$ and [illegible]

Let $T \in [a, \infty)$. If $T = a$, then $x(T) < y(T)$. Otherwise, assume $x(t) < y(t)$ for all $t \in [a, T)$. Then

$$x(T) \leq f(T) + \int_a^T K(s, T, x(s))ds$$

$$< f(T) + \int_a^T K(s, T, y(s))ds \leq y(T).$$

□

Example 2 (Comparison Theorem for Delay-Differential Equations):

Theorem: *Let $J = [t_0, \infty)$, $f \in C[J \times I\!R, I\!R]$, $x, y \in C^1[I\!R, I\!R]$, y be non-decreasing, and $f(t, x)$ be nondecreasing in x for each* [illegible]

$\alpha \in (0, \infty)$

i) $x'(t) \leq f(t, x(t - \alpha)) \quad \forall t \geq t_0$,

ii) [illegible]

iii) $x(t) \le y(t)$, $\forall t \in [t_0 - \alpha, t_0]$

then

[illegible]

Let $T \in [t_0, \infty)$. If $T = t_0$, then $x(T) \le y(T)$. Otherwise, assume

[illegible]

$$x'(s) \le f(s, x(s - \alpha)) \le f(s, y(s - \alpha)) \le f(s, y(s)) \le y'(s)$$

[illegible]

and hence $x'(t) \le y'(t)$ for all $t \in [t_0, T]$. Thus, $\int_{t_0}^{T} x'(s)ds \le \int_{t_0}^{T} y'(s)ds$, and so $x(T) - x(t_0) \le y(T) - y(t_0)$. By iii), $x(T) \le y(T)$. □

The final example establishes the asymptotic behavior of solutions to an integral inequality, given that the kernel is dominated by a function of a particular form.

Example 3: Let g be positive and continuous on $[0, \infty) \times (0, \infty)$ and $k > 0$. [illegible]

$$x(t) \le \int_0^t x(t-a) g(a,t) da + \frac{1}{2} k^{t+1}, \quad t \in [0, \infty)$$

are of exponential order as $t \to \infty$ provided $g(a,t) \le \frac{k^a}{2t}$ everywhere in $[0, \infty) \times (0, \infty)$. In particular, solutions decay exponentially if $0 < k < 1$.

We will show that $x(t) < k^{t+1}$ for all $t \in [0, \infty)$. [illegible]

Extend x to $\mathbb{R}$ by defining $x(t) = x(0)$ for all $t < 0$. Then the truth set $\{t \in \mathbb{R} \mid x(t) < k^{t+1}\}$ is open in $\mathbb{R}$ since x is continuous. Let $T \in [0, \infty)$. If $T = 0$, then $x(T) = x(0) \le \frac{1}{2}k < k = k^{T+1}$. Otherwise, assume that $x(t) < k^{t+1}$ whenever $t \in [0, T)$. Then

$$x(T) \le \int_0^T x(T-a) g(a,T) da + \frac{1}{2} k^{T+1}$$

$$< k^{T+1} \int_0^T \frac{1}{2T} da + \frac{1}{2} k^{T+1}$$

$$= k^{T+1}.$$

□

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